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A formulation of three-dimensional residual mean flow and wave activity flux applicable to equatorial waves

Takenari Kinoshita *

Integrated Science Data System Research Laboratory,

National Institute of Information and Communications Technology, Tokyo 184-8795, Japan.

AND

Kaoru Sato

Department of Earth and Planetary Science, The University of Tokyo, Tokyo, Japan.

* *Corresponding author address:* Takenari Kinoshita, Integrated Science Data System Research Laboratory, National Institute of Information and Communications Technology, Tokyo 184-8795, Japan.

E-mail: kinoshita@nict.go.jp

ABSTRACT

The large-scale waves that are known to be trapped around the equator are called equatorial waves. The equatorial waves cause mean zonal wind acceleration related to quasi-biennial and semiannual oscillations. The interaction between equatorial waves and the mean wind has been studied by using the transformed Eulerian-Mean (TEM) equations in the meridional cross section. However, to examine the three-dimensional (3D) structure of the interaction, the 3D residual mean flow and wave activity flux for the equatorial waves are needed. The 3D residual mean flow is expressed as the sum of the Eulerian-mean flow and the Stokes drift. The present study derives a formula that is approximately equal to the 3D Stokes drift for equatorial waves on the equatorial beta-plane (EQSD). The 3D wave activity flux for equatorial waves whose divergence corresponds to the wave forcing is also derived using the EQSD. It is shown that the meridionally integrated 3D wave activity flux for equatorial waves is proportional to the group velocity of equatorial waves.

1. Introduction

Equatorial waves are atmospheric waves whose amplitude is maximized at the equator and decreases exponentially as the distance of the waves from the equator increases. Their dispersion relation and spatial structure were studied theoretically by Matsuno (1966). He derived eigenmodes of equatorial waves by using a plane-wave assumption in the time and zonal directions on the linear shallow-water equation. The equatorial waves that were discovered by Wallace and Kousky (1968) and Yanai and Maruyama (1966) from radiosonde observation were identified as Kelvin waves and Rossby-gravity waves, respectively.

On the other hand, quasi-biennial and semiannual oscillations (QBO and SAO) of the zonal wind exist in the equatorial stratosphere. Previous studies have shown that these oscillations are driven by atmospheric waves. The relation between the waves and the zonal-mean zonal wind can be diagnosed by the transformed Eulerian-Mean (TEM) equations that were derived by Andrews and McIntyre (1976, 1978). The residual mean flow is expressed as the sum of the Eulerian-mean flow and the Stokes drift under the small amplitude assumption, and is approximately equal to the zonal-mean Lagrangian-mean flow when the wave is linear, steady, and adiabatic and when no dissipation occurs. The Eliassen-Palm (EP) flux is equal to the product of the group velocity and the wave activity density under the Wentzel-Kramers-Brillouin (WKB) approximation and is a useful physical quantity for describing the wave propagation (Edmon et al. 1980). The residual mean flow and the zonal-mean zonal wind acceleration are related to the divergence of EP flux in the zonal momentum equation. When there are no critical levels, the divergence of the EP flux is zero for linear, steady, and conservative waves. Under such conditions, the waves neither drive the residual mean flow nor accelerate the zonal-mean zonal wind. This is known as the non-acceleration theorem (Eliassen and Palm 1961; Charney and Drazin 1961). In studies using the TEM equations and equatorial wave theory, it was shown that the QBO is mainly driven by gravity waves, equatorial Kelvin waves, and Rossby-gravity waves (e.g., Sato and Dunkerton 1997; Haynes 1998; Baldwin et al. 2001), and it is recognized that the SAO is mainly driven by gravity waves, equatorial Kelvin waves and extratropical Rossby waves (e.g., Hirota 1980, Holton and Wehrbein 1980, Hitchman and Leovy 1988, Sassi and Garcia 1997, Antonita et al. 2007).

Kawatani et al. (2010) recently investigated the zonal variation of wave forcing associated with the equatorial Kelvin waves and inertia-gravity waves using the 3D wave activity flux derived by Miyahara (2006). They showed that this variation in the stratosphere results from zonal variation of the wave sources and from the vertically sheared zonal winds associated with the Walker circulation, depending on the phase of the QBO.

However, it is not clear whether this 3D wave activity flux can describe the equatorial Kelvin waves correctly because this flux is only applicable to inertia-gravity waves, not to equatorial waves. Although Kinoshita and Sato (2013a,b) newly formulated the 3D wave activity flux on the primitive equations, their formulae are not applicable in the equator region. The present study formulates the 3D residual mean flow and wave activity flux applicable to equatorial waves and shows that these formulae are applicable to the equatorial Kelvin waves.

The paper is arranged as follows. In section 2, the 3D Stokes drift (EQSD) is derived from its definition for the equatorial beta-plane equations. The 3D wave activity flux for equatorial waves (3D-EQW-flux) is formulated by using the EQSD in section 3. It is shown that the 3D-EQW-flux divergence corresponds to the equatorial wave forcing to the mean flow. It is also shown that the meridional integral of the 3D-EQW-flux accords with a product of the group velocity and the meridional integral of the wave activity density for equatorial waves. Moreover, we investigate the 3D wave-energy equation for equatorial waves. A summary and concluding remarks are given in section 4.

2. The time-mean 3D Stokes drift applicable to equatorial waves

When small-amplitude perturbations in the slowly varying background horizontal flow and weak background wind shear are assumed, the perturbation equations on the equatorial beta-plane are given as follows.

$$\bar{D}u' - \beta y v' + \Phi'_x = 0, \tag{2.1a}$$

$$\bar{D}v' + \beta y u' + \Phi'_y = 0, \tag{2.1b}$$

$$u'_x + v'_y + \rho_0^{-1}(\rho_0 w')_z = 0, \tag{2.1c}$$

$$\bar{D}\Phi'_z + N^2 w' = 0, \tag{2.1d}$$

and

$$\bar{D} \equiv \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}, \tag{2.1e}$$

where z is the log-pressure height, u, v, w, are zonal, meridional, and vertical velocities, respectively, ρ_0 is the basic density, Φ is the geopotential, N^2 is the buoyancy frequency squared, which expresses static stability, $\beta \equiv 2\Omega a^{-1}$ is the beta effect, Ω is the earth's rotation rate, a is the mean radius of the earth, the suffixes x, y, z denote the partial derivatives, and we assume that the time-mean vertical velocity, and the nonconservative and diabatic terms are negligible. The over bar ($^-$) and the prime ($'$) express the time mean and its deviation, respectively. For a perturbation, a form of plane wave is considered;

$$A' = \hat{A}(y)e^{z/2H}\exp[i(kx + mz - \omega t)], \tag{2.2a}$$

where A' is the arbitrary perturbation, H is the scale height, k, m are zonal and vertical wavenumbers, respectively, and ω is the ground-based angular frequency. It should be noted that the amplitudes of perturbations are constant in the time scale of wave phase change and vary in the time scale for the background state. Basic density is expressed as

$$\rho_0 = \rho_s \exp(-z/H), \tag{2.2b}$$

where ρ_s is a surface density. The zonal, meridional, and vertical parcel displacements $(\xi',\ \eta',\ \zeta')$ satisfy the following relations as

$$\bar{D}\xi' = u', \quad \bar{D}\eta' = v', \quad \bar{D}\zeta' = w'. \tag{2.3}$$

The time-mean Stokes drift is given in the following using the parcel displacements (2.3) and perturbation wind velocities.

$$\bar{u}^S = (\overline{\xi' u'})_x + (\overline{\eta' u'})_y + \rho_0^{-1}(\rho_0 \overline{\zeta' u'})_z = (\overline{\eta' u'})_y + \rho_0^{-1}(\rho_0 \overline{\zeta' u'})_z, \tag{2.4a}$$

$$\bar{v}^S = (\overline{\xi' v'})_x + (\overline{\eta' v'})_y + \rho_0^{-1}(\rho_0 \overline{\zeta' v'})_z = -(\overline{\eta' u'})_x + \rho_0^{-1}(\rho_0 \overline{\zeta' v'})_z, \tag{2.4b}$$

and

$$\bar{w}^S = (\overline{\xi' w'})_x + (\overline{\eta' w'})_y + \rho_0^{-1}(\rho_0 \overline{\zeta' w'})_z = -(\overline{\zeta' u'})_x - (\overline{\zeta' v'})_y. \tag{2.4c}$$

Here, it should be noted that the deformations on the second equal sign of each equation are made by using the relations $(\overline{\xi' u'}) = (\overline{\eta' v'}) = (\overline{\zeta' w'}) = 0, (\overline{\xi' v'}) = -(\overline{\eta' u'}), (\overline{\xi' w'}) = -(\overline{\zeta' u'})$, and $(\overline{\eta' w'}) = -(\overline{\zeta' v'})$ under the assumption that the time-mean wind shear is small.

In the next section, the EQSD for equatorial Kelvin waves, Rossby-gravity waves, and other-types of equatorial waves are formulated from the definition (2.4).

a. The 3D Stokes drift for equatorial Kelvin waves

For equatorial Kelvin waves, the meridional component of perturbation wind velocity and that of parcel displacement vanish. Thus, EQSD has only the zonal and vertical components. When (2.2) is substituted into (2.1d) and (2.3), the vertical parcel displacement ζ' is written in terms of Φ' as

$$\zeta' = -\frac{\Phi'_z}{N^2} = -\frac{(im + 1/2H)}{N^2}\Phi'. \tag{2.5}$$

Using (2.5) enables $\overline{\zeta' u'}$ to be expressed as

$$\overline{\zeta' u'} = -\frac{\overline{u' \Phi'_z}}{N^2}. \tag{2.6}$$

Thus, EQSD for the equatorial Kelvin wave is formulated in the following.

$$\bar{u}^S_{(\mathrm{Kl})} = -\frac{1}{\rho_0}\left(\frac{\rho_0 \overline{u'\Phi'_z}}{N^2}\right)_z, \tag{2.7a}$$

and

$$\bar{w}^S_{(\mathrm{Kl})} = \left(\frac{\overline{u'\Phi'_z}}{N^2}\right)_x, \tag{2.7b}$$

where the subscript (Kl) is used to distinguish other equatorial waves.

Next, the difference between $\bar{\mathbf{u}}^S_{(\mathrm{Kl})}$ and other 3D Stokes drifts is examined in terms of $\overline{S} \equiv \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} - \frac{\overline{\Phi_z'^2}}{N^2}\right)$, and $\overline{S_{(p)}} \equiv \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} - \frac{\overline{u'\Phi'_y}}{f} + \frac{\overline{v'\Phi'_x}}{f}\right)$ (Kinoshita and Sato 2013a,b). While the term $\overline{S}$ becomes equal to $f\overline{u'\eta'}$ and is included in 3D Stokes drift for inertia-gravity waves, the term $\overline{S_{(p)}}$ becomes equal to $f\overline{u'\eta'}$ and is included in the one applicable to both Rossby waves and gravity waves. Note that $\eta' = -i\hat{\omega}^{-1}v'$ becomes equal to zero for equatorial Kelvin waves. From zonal momentum, continuity, and thermodynamic equations, a polarization and dispersion relations for equatorial Kelvin waves are expressed as follows (Andrews et al. 1987).

$$u' = \frac{k}{\hat{\omega}}\Phi', \tag{2.8}$$

$$\hat{\omega}^2 = N^2 k^2/\tilde{m}^2, \tag{2.9}$$

where we use $\bar{D} = -i\hat{\omega}$ and assume that $\hat{\omega}$ is independent of the latitude. From zonal and meridional momentum equation, the geopotential of equatorial Kelvin waves is expressed as

$$\Phi' = \hat{\Phi}_0 e^{z/2H}\exp(-\beta k y^2/2\hat{\omega})\exp[i(kx + mz - \omega t)], \tag{2.10}$$

where $\hat{\Phi}_0$ is constant. Using (2.8) and (2.10), the term $\overline{S}$ is written in terms of Φ' as

$$\overline{S} = \frac{1}{2}\left(\overline{u'^2} - \frac{\overline{\Phi_z'^2}}{N^2}\right) = \frac{1}{2}\left(\frac{k^2}{\hat{\omega}^2} - \frac{\tilde{m}^2}{N^2}\right)\overline{\Phi'^2} = 0. \tag{2.11}$$

Similarly, the term $\overline{S_{(p)}}$ becomes

$$\overline{S_{(p)}} = \frac{1}{2}\left(\overline{u'^2} - \frac{\overline{u'\Phi'_y}}{\beta y}\right) = \frac{1}{2}\left(\frac{k^2}{\hat{\omega}^2} + \frac{\beta k^2 y}{\hat{\omega}^2 \beta y}\right)\overline{\Phi'^2} = \frac{k^2}{\hat{\omega}^2}\overline{\Phi'^2}. \tag{2.12}$$

Thus, 3D Stoke drift for equatorial Kelvin waves is equal to that applicable only to inertia-gravity waves (Kinoshita et al. 2010), not equal to that derived by Kinoshita and Sato (2013a,b).

b. The 3D Stokes drift for other types of equatorial waves

For waves having non-zero meridional components of perturbation wind velocity, slightly complex manipulation is needed to relate to the perturbation meridional velocity and other perturbation physical quantities. The dispersion relation for equatorial waves and the solution for $\hat{v}(y)$ are expressed as follows.

$$\frac{\tilde{m}^2\hat{\omega}^2}{N^2} - k^2 - \frac{k\beta}{\hat{\omega}} = (2n+1)\frac{\beta|\tilde{m}|}{N}, \tag{2.13a}$$

$$\hat{v} = \hat{v}_0 \exp{(-Y^2/2)} H_n(Y), \tag{2.13b}$$

and

$$Y \equiv \sqrt{\frac{\beta|\tilde{m}|}{N}} y, \tag{2.13c}$$

where $\tilde{m}^2 = m^2 + 1/4H^2$, $H_n(Y)$ are the Hermite polynomials and $\hat{v}_0$ is constant. Substituting (2.13b) into (2.1) and using the identities $dH_n(Y)/dY = 2nH_{n-1}(Y)$, and$H_{n+1} = 2YH_n(Y) - 2nH_{n-1}(Y)$ make it possible to show that

$$\hat{u} = i\hat{v}_0 \exp{(-Y^2/2)} \sqrt{\beta|\tilde{m}|N} \left(\frac{1/2H_{n+1}(Y)}{|\tilde{m}|\hat{\omega} - Nk} + \frac{nH_{n-1}(Y)}{|\tilde{m}|\hat{\omega} + Nk} \right), \tag{2.14a}$$

and

$$\hat{\Phi} = i\hat{v}_0 \exp{(-Y^2/2)} \sqrt{\frac{\beta N^3}{|\tilde{m}|}} \left(\frac{1/2H_{n+1}(Y)}{|\tilde{m}|\hat{\omega} - Nk} - \frac{nH_{n-1}(Y)}{|\tilde{m}|\hat{\omega} + Nk} \right), \tag{2.14b}$$

(Andrews et al. 1987). It is noted that the latitudinal scale of the background fields is larger than an equatorial radius of deformation $\sqrt{\frac{N}{\beta|\tilde{m}|}}$. First, by using (2.3), (2.14a) and (2.13b), $\overline{\eta' u'}$ is expressed in terms of $\hat{v}_0$ as follows.

$$\overline{\eta' u'} = \frac{\hat{v}_0^2}{2} \exp{(-Y^2)} \frac{\sqrt{\beta|\tilde{m}|N}}{\hat{\omega}} \left(\frac{1/2H_{n+1}(Y)}{|\tilde{m}|\hat{\omega} - Nk} + \frac{nH_{n-1}(Y)}{|\tilde{m}|\hat{\omega} + Nk} \right) H_n(Y). \tag{2.15}$$

Similarly, $\frac{1}{2}\overline{u'^2}$, $\frac{1}{2}\overline{v'^2}$, and a potential energy are written in the following.

$$\begin{aligned}
\frac{1}{2}\overline{u'^2} &= \frac{\hat{v}_0^2}{4}\exp(-Y^2)\beta|\tilde{m}|N\left(\frac{1/2H_{n+1}(Y)}{|\tilde{m}|\hat{\omega}-Nk}+\frac{nH_{n-1}(Y)}{|\tilde{m}|\hat{\omega}+Nk}\right)^2, && (2.16a)\\
\frac{1}{2}\overline{v'^2} &= \frac{\hat{v}_0^2}{4}\exp(-Y^2)H_n^2(Y), && (2.16b)
\end{aligned}$$

and

$$\frac{1}{2}\frac{\overline{\Phi_z'^2}}{N^2} = \frac{\hat{v}_0^2}{4}\exp(-Y^2)\beta|\tilde{m}|N\left(\frac{1/2H_{n+1}(Y)}{|\tilde{m}|\hat{\omega}-Nk}-\frac{nH_{n-1}(Y)}{|\tilde{m}|\hat{\omega}+Nk}\right)^2. \qquad (2.16c)$$

The meridional derivative of the difference between (2.16a) and (2.16c), and the derivative of (2.16b) are respectively expressed as

$$\begin{aligned}
\frac{d}{dy}\frac{1}{2}\left(\overline{u'^2}-\frac{\overline{\Phi_z'^2}}{N^2}\right) &= \sqrt{\frac{\beta|\tilde{m}|}{N}}\beta|\tilde{m}|N\hat{v}_0^2\frac{d}{dY}\left(\frac{\exp(-Y^2)}{4}\frac{2nH_{n+1}(Y)H_{n-1}(Y)}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}\right)\\
&= \sqrt{\frac{\beta|\tilde{m}|}{N}}\beta|\tilde{m}|N\hat{v}_0^2\exp(-Y^2)\\
&\times \frac{n(n+1)H_n(Y)H_{n-1}(Y)-\frac{1}{2}nH_{n+1}(Y)H_n(Y)}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}, && (2.17a)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dy}\frac{1}{2}\overline{v'^2} &= \sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2\frac{d}{dY}\left(\frac{\exp(-Y^2)}{4}H_n^2(Y)\right)\\
&= \sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2\frac{\exp(-Y^2)}{2}\left(-YH_n^2(Y)^2+2nH_n(Y)H_{n-1}(Y)\right)\\
&= \sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2\frac{\exp(-Y^2)}{2}\\
&\times \left(-\frac{1}{2}H_{n+1}(Y)H_n(Y)+nH_n(Y)H_{n-1}(Y)\right). && (2.17b)
\end{aligned}$$

Hereafter, the notation (Y) is omitted from $H_n(Y)$. From the difference between (2.17b) and (2.17a), the following relation is obtained.

$$
\begin{aligned}
&\frac{d}{dy}\frac{1}{2}\left(\overline{u'^2}-\overline{v'^2}-\frac{\overline{\Phi_z'^2}}{N^2}\right)\\
&=\sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2 e^{-Y^2}\\
&\times\left(\beta|\tilde{m}|N\frac{n(n+1)H_nH_{n-1}-\frac{1}{2}nH_{n+1}H_n}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}+\frac{1}{4}H_nH_{n+1}-\frac{n}{2}H_nH_{n-1}\right)\\
&=\sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2 e^{-Y^2}\left\{\frac{[-2n\beta|\tilde{m}|N+\beta kN^2\hat{\omega}^{-1}+(2n+1)\beta|\tilde{m}|N]H_{n+1}H_n}{4(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)}\right\}\\
&+\sqrt{\frac{\beta|\tilde{m}|}{N}}\hat{v}_0^2 e^{-Y^2}\left(\frac{\{4n(n+1)\beta|\tilde{m}|N-2n[\beta kN^2\hat{\omega}^{-1}+(2n+1)\beta|\tilde{m}|N]\}H_nH_{n-1}}{4(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)}\right)\\
&=\frac{\hat{v}_0^2e^{-Y^2}}{2}\frac{\beta\sqrt{\beta|\tilde{m}|N}}{\hat{\omega}}\left(\frac{\frac{1}{2}H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk}+\frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)H_n=\beta\overline{u'\eta'}.
\end{aligned}
\tag{2.18}
$$

Next, $\overline{\zeta' u'}$ and $\overline{\zeta' v'}$ are deformed in the same way as (2.6)

$$
\overline{\zeta' u'}=-\frac{\overline{u'\Phi_z'}}{N^2},\quad \overline{\zeta' v'}=-\frac{\overline{v'\Phi_z'}}{N^2}. \tag{2.19}
$$

Thus, the 3D Stokes drift for equatorial waves (EQSD) is formulated as follows.

$$
\bar{u}^S_{(\mathrm{EQ})}=\left[\frac{1}{\beta}\frac{\partial}{\partial y}\frac{1}{2}\left(\overline{u'^2}-\overline{v'^2}-\frac{\overline{\Phi_z'^2}}{N^2}\right)\right]_y-\frac{1}{\rho_0}\left(\frac{\rho_0\overline{u'\Phi_z'}}{N^2}\right)_z, \tag{2.20a}
$$

$$
\bar{v}^S_{(\mathrm{EQ})}=-\left[\frac{1}{\beta}\frac{\partial}{\partial y}\frac{1}{2}\left(\overline{u'^2}-\overline{v'^2}-\frac{\overline{\Phi_z'^2}}{N^2}\right)\right]_x-\frac{1}{\rho_0}\left(\frac{\rho_0\overline{v'\Phi_z'}}{N^2}\right)_z, \tag{2.20b}
$$

and

$$
\bar{w}^S_{(\mathrm{EQ})}=\left(\frac{\overline{u'\Phi_z'}}{N^2}\right)_x+\left(\frac{\overline{v'\Phi_z'}}{N^2}\right)_y. \tag{2.20c}
$$

It is important that the EQSD (2.20) is also applicable to the Kelvin waves (n=-1) since $\overline{u'^2}=\overline{\Phi_z'^2}/N^2$ and $\overline{v'^2}=0$, and hence $\overline{\eta' u'}=0$. Thus, the EQSD (2.20) can be used for all types of equatorial waves. It should be noted that the advantage of EQSD (2.20) is to be derived without including parcel displacements which are hardly observed and to be composed of eddy covariances. This means that the EQSD is applicable to not only monochromatic waves but also all equatorially confined perturbations that are expressed with a superposition of sinusoidal waves.

3. A formulation of the 3D wave activity flux for equatorial waves

a. *The 3D residual mean flow and wave activity flux*

The time-mean zonal momentum equation on the equatorial beta-plane is given by

$$\bar{u}_t + \bar{u}_x\bar{u} + (\bar{u}_y - \beta y)\bar{v} + \bar{u}_z\bar{w} = -(\overline{u'^2})_x - (\overline{u'v'})_y - \rho_0^{-1}(\rho_0\overline{u'w'})_z. \quad (3.1)$$

By substituting (2.20) into (3.1) and using the assumption that the background wind shear is negligible, we obtain

$$\bar{u}_t - \beta y \overline{v}^*_{(\mathrm{EQ})} = -\rho_0^{-1}(\nabla \cdot \mathbf{F_1^{(EQ)}}), \quad (3.2)$$

where $\overline{v}^*_{(\mathrm{EQ})} = \bar{v} + \bar{v}^S_{(\mathrm{EQ})}$ is the meridional component of the 3D residual mean flow associated with forcing by equatorial waves, and $\mathbf{F_1^{(EQ)}} = (F_{11}^{(\mathrm{EQ})},\ F_{12}^{(\mathrm{EQ})},\ F_{13}^{(\mathrm{EQ})})$ is the 3D wave activity flux for equatorial waves:

$$F_{11}^{(\mathrm{EQ})} = \rho_0\left\{\overline{u'^2} - \frac{y}{2}\frac{\partial}{\partial y}\left(\overline{u'^2} - \overline{v'^2} - \frac{\overline{\Phi_z'^2}}{N^2}\right)\right\}, \quad (3.3a)$$

$$F_{12}^{(\mathrm{EQ})} = \rho_0(\overline{u'v'}) = 0, \quad (3.3b)$$

and

$$F_{13}^{(\mathrm{EQ})} = \rho_0\left(\overline{u'w'} - \beta y\frac{\overline{v'\Phi_z'}}{N^2}\right). \quad (3.3c)$$

It should be noted that (3.3b) vanishes since u' and v' are out of phase by 90 degrees. This 3D wave activity flux (3.3) is related to the wave forcing for the time-mean flow. In the next section, the relation between the 3D wave activity flux (3.3) and the group velocity of equatorial waves is examined.

b. *The relation between 3D wave activity flux and group velocity*

In the 2D TEM equation system, the meridionally integrated EP flux is equal to a product of the vertical group velocity and the meridionally integrated wave activity density (Andrews et al. 1987). It can be shown that the vertical component of 3D wave activity flux (3.3c) satisfies this relation, as in the following.

Using (2.1d) and (2.14) enables us to write $\overline{u'w'}$ included in (3.3c) in terms of $\hat{v}_0$ as

$$\overline{u'w'} = (-m\hat{\omega}\beta)\frac{\hat{v}_0 e^{-Y^2}}{2}\left\{\frac{(\frac{1}{2}H_{n+1})^2}{(|\tilde{m}|\hat{\omega}-Nk)^2} - \frac{(nH_{n-1})^2}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right\}. \tag{3.4}$$

The meridional integral of (3.4) is obtained by using the dispersion relation of equatorial waves (2.13a) and $\int_{-\infty}^{\infty} H_m(Y)H_n(Y)\exp(-Y^2)dY = \delta_{m,n}2^n n!\sqrt{\pi}$:

$$\begin{aligned}\int_{-\infty}^{\infty}\overline{u'w'}dy &= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}(-m\hat{\omega}\beta)2^{n-1}n!\sqrt{\pi}\left[\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2} - \frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right]\\ &= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}(-m\hat{\omega}\beta)2^{n-1}n!\sqrt{\pi}\frac{2\hat{\omega}k+\beta}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}.\end{aligned} \tag{3.5}$$

Similarly, $-\beta y\frac{\overline{v'\Phi'_z}}{N^2}$ and its meridional integral are given in the following.

$$\begin{aligned}-\beta y\frac{\overline{v'\Phi'_z}}{N^2} &= \frac{m\beta}{|\tilde{m}|}\frac{\hat{v}_0^2 e^{-Y^2}}{2}\sqrt{\frac{\beta|\tilde{m}|}{N}}y\left(\frac{\frac{1}{2}H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk} - \frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}-Nk}\right)H_n\\ &= \frac{m\beta}{|\tilde{m}|}\frac{\hat{v}_0^2 e^{-Y^2}}{2}\left[\frac{(\frac{1}{2}H_{n+1})^2}{|\tilde{m}|\hat{\omega}-Nk} - \frac{(nH_{n-1})^2}{|\tilde{m}|\hat{\omega}+Nk} + \frac{NknH_{n+1}H_{n-1}}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}\right],\end{aligned} \tag{3.6a}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}-\beta y\frac{\overline{v'\Phi'_z}}{N^2}dy &= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}\left(\frac{m\beta}{|\tilde{m}|}\right)2^{n-1}n!\sqrt{\pi}\left(\frac{n+1}{|\tilde{m}|\hat{\omega}-Nk} - \frac{n}{|\tilde{m}|\hat{\omega}+Nk}\right)\\ &= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}\left(\frac{m\beta}{|\tilde{m}|}\right)2^{n-1}n!\sqrt{\pi}\frac{(2n+1)Nk+|\tilde{m}|\hat{\omega}}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}\}.\end{aligned} \tag{3.6b}$$

Thus,

$$\int_{-\infty}^{\infty}F_{13}^{(\mathrm{EQ})}dy = \rho_0\sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}2^{n-1}n!\sqrt{\pi}\frac{-2\hat{\omega}^2km+(2n+1)\beta Nkm|\tilde{m}|^{-}1}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}. \tag{3.7}$$

On the other hand, the wave activity density and its meridional integral are written as

$$\frac{E^{(\mathrm{EQ})}}{\hat{C}_{(\mathrm{x})}} = \rho_0\frac{\hat{v}_0^2 e^{-Y^2}}{2}\frac{k}{\hat{\omega}}\left\{\beta|\tilde{m}|N\left[\frac{(\frac{1}{2}H_{n+1})^2}{(|\tilde{m}|\hat{\omega}-Nk)^2} + \frac{(nH_{n-1})^2}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right] + \frac{H_n^2}{2}\right\}, \tag{3.8a}$$

and

$$\int_{-\infty}^{\infty} \frac{E^{(\mathrm{EQ})}}{\hat{C}_{(\mathrm{x})}} dy = \rho_0 \sqrt{\frac{N}{\beta|\tilde{m}|}} \frac{\beta|\tilde{m}|Nk}{\hat{\omega}} \frac{\hat{v}_0^2}{2} 2^{n-1} n! \sqrt{\pi} \times \left[\frac{n+1}{(|\tilde{m}|\hat{\omega} - Nk)^2} + \frac{n}{(|\tilde{m}|\hat{\omega} + Nk)^2} + \frac{1}{\beta|\tilde{m}|N} \right] = \rho_0 \sqrt{\frac{N}{\beta|\tilde{m}|}} \hat{\omega} \frac{\hat{v}_0^2}{2} 2^{n-1} n! \sqrt{\pi} \frac{k}{\hat{\omega}} \frac{2|\tilde{m}|^2\hat{\omega}^2 + \hat{\omega}^{-1}N^2k\beta}{|\tilde{m}|^2\hat{\omega}^2 - N^2k^2}, \tag{3.8b}$$

where $E^{(\mathrm{EQ})} \equiv \frac{\rho_0}{2} \left(\overline{u'^2} + \overline{v'^2} + \frac{\overline{\Phi_z'^2}}{N^2} \right)$. The derivation of (3.5), (3.6b) and (3.8b) is given in the Appendix. The zonal and vertical group velocities of equatorial waves are expressed as

$$\hat{C}_{(\mathrm{gx})}^{(\mathrm{EQ})} = \frac{2\hat{\omega}N^2k + N^2\beta}{2|\tilde{m}|^2\hat{\omega}^2 + \hat{\omega}^{-1}N^2k\beta}, \tag{3.9a}$$

and

$$\hat{C}_{(\mathrm{gz})}^{(\mathrm{EQ})} = \frac{-2\hat{\omega}^3 m + (2n+1)\hat{\omega}N\beta}{2|\tilde{m}|^2\hat{\omega}^2 + \hat{\omega}^{-1}N^2k\beta}. \tag{3.9b}$$

Dividing (3.7) by (3.8b) yields

$$\frac{\int_{-\infty}^{\infty} F_{13}^{(\mathrm{EQ})} dy}{\int_{-\infty}^{\infty} \frac{E^{(\mathrm{EQ})}}{\hat{C}_{(\mathrm{x})}} dy} = \frac{k}{\hat{\omega}} \frac{-2\hat{\omega}^2 km + (2n+1)\beta Nkm|\tilde{m}|^{-1}}{2|\tilde{m}|^2\hat{\omega}^2 + \hat{\omega}^{-1}N^2k\beta} = \hat{C}_{(\mathrm{gz})}^{(\mathrm{EQ})}. \tag{3.10}$$

Thus, the meridional integral of the vertical component of 3D wave activity flux for equatorial waves (3.3c) is proportional to the vertical group velocity.

Next, it is shown that the meridional integral of the zonal component of 3D wave activity flux for equatorial waves (3.3a) accords with a product of the zonal group velocity and the meridionally integrated wave activity density. Using (2.13b) and (2.14), $F_{11}^{(\mathrm{EQ})}$ and its meridional integral are written in terms of $\hat{v}_0$ as

$$
\begin{aligned}
F_{11}^{(\mathrm{EQ})} &= \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \beta |\tilde{m}| N \left(\frac{\frac{1}{2} H_{n+1}}{|\tilde{m}|\hat{\omega} - Nk} + \frac{n H_{n-1}}{|\tilde{m}|\hat{\omega} + Nk} \right)^2 \\
&- \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta y \sqrt{\beta |\tilde{m}| N}}{\hat{\omega}} \left(\frac{\frac{1}{2} H_{n+1}}{|\tilde{m}|\hat{\omega} - Nk} + \frac{n H_{n-1}}{|\tilde{m}|\hat{\omega} + Nk} \right) H_n \\
&= \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta N}{\hat{\omega}} |\tilde{m}|\hat{\omega} \left(\frac{\frac{1}{2} H_{n+1}}{|\tilde{m}|\hat{\omega} - Nk} + \frac{n H_{n-1}}{|\tilde{m}|\hat{\omega} + Nk} \right)^2 \\
&- \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta N}{\hat{\omega}} \left(\frac{\frac{1}{2} H_{n+1}}{|\tilde{m}|\hat{\omega} - Nk} + \frac{n H_{n-1}}{|\tilde{m}|\hat{\omega} + Nk} \right) (\frac{1}{2} H_{n+1} + n H_{n-1}) \\
&= \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta N}{\hat{\omega}} \\
&\times \left[\frac{(|\tilde{m}|\hat{\omega} - |\tilde{m}|\hat{\omega} + Nk)(\frac{1}{2} H_{n+1})^2}{(|\tilde{m}|\hat{\omega} - Nk)^2} + \frac{(|\tilde{m}|\hat{\omega} - |\tilde{m}|\hat{\omega}) H_{n+1} H_{n-1}}{|\tilde{m}|^2 \hat{\omega}^2 - N^2 k^2} \right] \\
&+ \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta N}{\hat{\omega}} \frac{(|\tilde{m}|\hat{\omega} - |\tilde{m}|\hat{\omega} + Nk)(n H_{n-1})^2}{(|\tilde{m}|\hat{\omega} + Nk)^2} \\
&= \rho_0 \frac{\hat{v}_0^2 e^{-Y^2}}{2} \frac{\beta N^2 k}{\hat{\omega}} \left[\frac{\frac{1}{4} H_{n+1}^2}{(|\tilde{m}|\hat{\omega} - Nk)^2} - \frac{n^2 H_{n-1}^2}{(|\tilde{m}|\hat{\omega} + Nk)^2} \right]. \qquad (3.11a)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{\infty} F_{11}^{(\mathrm{EQ})} dy &= \rho_0 \sqrt{\frac{N}{\beta |\tilde{m}|}} \frac{\hat{v}_0^2}{2} 2^{n-1} n! \sqrt{\pi} \frac{\beta N^2 k}{\hat{\omega}} \left[\frac{n+1}{(|\tilde{m}|\hat{\omega} - Nk)^2} - \frac{n}{(|\tilde{m}|\hat{\omega} + Nk)^2} \right] \\
&= \rho_0 \sqrt{\frac{N}{\beta |\tilde{m}|}} \frac{\hat{v}_0^2}{2} 2^{n-1} n! \sqrt{\pi} \frac{k}{\hat{\omega}} \frac{2\hat{\omega} N^2 k + N^2 \beta}{|\tilde{m}|^2 \hat{\omega}^2 - N^2 k^2}. \qquad (3.11b)
\end{aligned}
$$

Dividing (3.11) by (3.8b) yields

$$
\frac{\int_{-\infty}^{\infty} F_{11}^{(\mathrm{EQ})} dy}{\int_{-\infty}^{\infty} \frac{E^{(\mathrm{EQ})}}{\hat{C}_{(\mathrm{x})}} dy} = \frac{2\hat{\omega} N^2 k + N^2 \beta}{2 |\tilde{m}|^2 \hat{\omega}^2 + \hat{\omega}^{-1} N^2 k \beta} = \hat{C}_{(\mathrm{gx})}^{(\mathrm{EQ})}. \qquad (3.12)
$$

These results indicate that the 3D wave activity flux (3.3) can describe the propagation of equatorial waves. It should be noted that the terms proportional to the group velocities are not the 3D wave activity flux (3.3) but its meridional integral. This is similar to the case of EP flux for equatorial waves.

c. The wave-energy equation for equatorial waves

This section examines how the 3D wave activity flux for equatorial waves is related to the wave activity density after the wave-energy equation is derived. In this derivation, it is assumed that the time-mean wind shear is negligible.

First, taking $u'\times$(2.1a)$+v'\times$(2.1b)$+\Phi'_z/N^2\times$(2.1d) and using the time mean yield

$$\bar{D}E^{(\mathrm{EQ})} + \rho_0^{-1}(\nabla\cdot\rho_0\overline{\mathbf{u}'\Phi'}) = 0. \tag{3.13}$$

This equation (3.13) is regarded as the 3D wave-energy equation for equatorial waves. Note that $\overline{v'\Phi'}$ vanishes since v' and Φ' are out of phase by 90 degrees.

Next, by using (2.13b) and (2.14), $\overline{u'\Phi'}$ can be written in terms of $\hat{v}_0$ as

$$\overline{u'\Phi'} = \rho_0\frac{\hat{v}_0^2e^{-Y^2}}{2}\beta N^2\left[\frac{\frac{1}{4}H_{n+1}^2}{(|\tilde{m}|\hat{\omega}-Nk)^2} - \frac{n^2H_{n-1}^2}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right]. \tag{3.14}$$

From (3.11a) and (3.14)

$$F_{11}^{(\mathrm{EQ})} = \rho_0\frac{\hat{\omega}}{k}\overline{u'\Phi'}. \tag{3.15}$$

Similarly,

$$F_{13}^{(\mathrm{EQ})} = \rho_0\frac{\hat{\omega}}{k}\overline{w'\Phi'}. \tag{3.16}$$

From (3.13), (3.15), and (3.16),

$$\bar{D}\frac{E^{(\mathrm{EQ})}}{\hat{C}_{(\mathrm{x})}} + \rho_0^{-1}(\nabla\cdot\mathbf{F}_\mathbf{1}^{(\mathrm{EQ})}) = 0. \tag{3.17}$$

This equation (3.17) is regarded as the generalized Eliassen-Palm relation for equatorial waves under the slowly varying time-mean flow assumption. Note that the equations (3.2) and (3.17) express the wave-mean flow interaction as is consistent with equations (3.5a), (5.5a) and (5.7) in Andrews and McIntyre (1976). It should be noted that these relations (3.13) and (3.17) are obtained without using the meridional integral, unlike the results of section 3b.

4. Concluding remarks

In this study, the 3D Stokes drift was formulated from its definition for equatorial beta-plane equations (EQSD) when the slowly varying background field and small amplitude perturbations are assumed. The EQSD is applicable to all equatorial waves. The 3D wave activity flux (3D EQW-flux) was formulated by substituting the EQSD into the time-mean zonal momentum equation. These expressions are derived using the time mean and are phase independent.

Next, it was shown that the latitudinal integral of 3D EQW-flux accords with a product of the group velocity and the latitudinally integrated wave activity density in both zonal and vertical directions. This is an extension of the relation for the Eliassen-Palm flux on the 2D TEM equations. The present study also derived the 3D wave-energy equation for equatorial waves.

As it is shown that $\bar{\mathbf{u}}^S_{(\mathrm{Kl})}$ becomes equal to the 3D Stokes drift for gravity waves in section 2.a, we compare $F_{11}^{(\mathrm{EQ})}$ and other 3D wave activity flux. The result shows that the meridional integral of $F_{11}^{(\mathrm{EQ})}$ is equal to that of 3D wave activity flux applicable to inertia-gravity waves (Miyahara 2006; Kinoshita et al. 2010) $\frac{\rho_0}{2}\left(\overline{u'^2} - \overline{v'^2} + \frac{\overline{\Phi_z'^2}}{N^2}\right)$. The detail is written in Appendix.

The EQSD and 3D EQW-flux are partly different from the 3D Stokes drift and wave activity flux that are applicable to both gravity waves and Rossby waves (Kinoshita and Sato 2013a,b). The difference is due to assumptions of waves. Kinoshita and Sato (2013a,b) assumes waves having meridional wavenumbers, and the term $\overline{u'\eta'}$ is reduced to $\frac{1}{2f}\left(\overline{u'^2} + \overline{v'^2} - \frac{\overline{u'\Phi_y'}}{f} + \frac{\overline{v'\Phi_x'}}{f}\right)$. On the other hand, this study assumes waves whose amplitude is damped in the meridional direction, and the term $\overline{u'\eta'}$ is reduced to $\frac{1}{\beta}\frac{\partial}{\partial y}\frac{1}{2}\left(\overline{u'^2} - \overline{v'^2} - \frac{\overline{\Phi_z'^2}}{N^2}\right)$. Similar manipulations may be needed for a case of tidal waves whose meridional structures have some nodes. Thus, the 3D TEM equations applicable to tidal waves need to be derived.

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APPENDIX A

Derivation of (3.5), (3.6b), and (3.8b)

The deformation of the first line of (3.5) is made by using (2.13c) and $\int_{-\infty}^{\infty} H_m(Y)H_n(Y)\exp(-Y^2)dY = \delta_{m,n}2^n n!\sqrt{\pi}$:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \overline{u'w'}dy \\
&= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}(-m\hat{\omega}\beta)\int_{-\infty}^{\infty}\left\{\frac{\frac{1}{4}H_{n+1}^2}{(|\tilde{m}|\hat{\omega}-Nk)^2}-\frac{n^2H_{n-1}^2}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right\}e^{-Y^2}dY \\
&= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}(-m\hat{\omega}\beta)\left\{\frac{2^{n-1}(n+1)!}{(|\tilde{m}|\hat{\omega}-Nk)^2}-\frac{2^{n-1}n\times n!}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right\}\sqrt{\pi} \\
&= \sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}(-m\hat{\omega}\beta)2^{n-1}n!\sqrt{\pi}\left\{\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2}-\frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2}\right\}. \qquad \text{(A1)}
\end{aligned}
$$

The deformation of the first line of (3.6b) and from the first to the second line of (3.8b) are also made in a similar way.

Next, by using the dispersion relation of equatorial waves (2.13a), the part $\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2} - \frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2}$ included in (3.5) can be expressed as follows.

$$
\begin{aligned}
&\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2}-\frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2} \\
&= \frac{4n|\tilde{m}|\hat{\omega}Nk+(|\tilde{m}|\hat{\omega}+Nk)^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{4n|\tilde{m}|\hat{\omega}Nk+(|\tilde{m}|\hat{\omega}+Nk)^2+2|\tilde{m}|\hat{\omega}Nk-2|\tilde{m}|\hat{\omega}Nk}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{2|\tilde{m}|\hat{\omega}Nk(2n+1)+|\tilde{m}|^2\hat{\omega}^2+N^2k^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{2|\tilde{m}|\hat{\omega}Nk\left(\frac{|\tilde{m}|\hat{\omega}^2}{\beta N}-\frac{Nk^2}{\beta|\tilde{m}|}-\frac{Nk}{|\tilde{m}|\hat{\omega}}\right)+|\tilde{m}|^2\hat{\omega}^2+N^2k^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{2\hat{\omega}k\beta^{-1}(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)-2N^2k^2+|\tilde{m}|^2\hat{\omega}^2+N^2k^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{2\hat{\omega}k\beta^{-1}+1}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)}. \qquad \text{(A2)}
\end{aligned}
$$

Similarly, the parts $\frac{n+1}{|\tilde{m}|\hat{\omega}-Nk} - \frac{n}{|\tilde{m}|\hat{\omega}+Nk}$ in (3.6b), and $\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2} + \frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2} + \frac{1}{\beta|\tilde{m}|N}$ in (3.8b) are reduced in the following.

$$\frac{n+1}{|\tilde{m}|\hat{\omega}-Nk}-\frac{n}{|\tilde{m}|\hat{\omega}+Nk} = \frac{(|\tilde{m}|\hat{\omega}+Nk)(n+1)-(|\tilde{m}|\hat{\omega}-Nk)n}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2} = \frac{(2n+1)Nk+|\tilde{m}|\hat{\omega}}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}, \tag{A3a}$$

and

$$\begin{aligned}
&\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2} + \frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2} + \frac{1}{\beta|\tilde{m}|N} \\
&= \frac{2(|\tilde{m}|^2\hat{\omega}^2+N^2k^2)n + (|\tilde{m}|\hat{\omega}+Nk)^2 + \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2}{\beta|\tilde{m}|N}}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{(|\tilde{m}|^2\hat{\omega}^2+N^2k^2)(2n+1) - |\tilde{m}|^2\hat{\omega}^2 - N^2k^2 + (|\tilde{m}|\hat{\omega}+Nk)^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&+ \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)\{k\hat{\omega}^{-1}\beta N^2+(2n+1)\beta|\tilde{m}|N\}}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{2|\tilde{m}|^2\hat{\omega}^2(2n+1) + 2|\tilde{m}|\hat{\omega}Nk + \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)\beta N^2k}{\beta|\tilde{m}|N\hat{\omega}}}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)(2|\tilde{m}|^2\hat{\omega}^2+k\hat{\omega}^{-1}\beta N^2)}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
&= \frac{N}{\beta|\tilde{m}|}\frac{2|\tilde{m}|^2\hat{\omega}^2N^{-2}+k\hat{\omega}^{-1}\beta}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}.
\end{aligned} \tag{A3b}$$

APPENDIX B

Meridional integral of 3D wave activity flux applicable to inertia-gravity waves

The meridional integral of $\frac{\rho_0}{2}\left(\overline{u'^2}-\overline{v'^2}+\frac{\overline{\Phi_z'^2}}{N^2}\right)$ is expressed as

$$\begin{aligned}
\int_{-\infty}^{\infty} & \frac{\rho_0}{2}\left(\overline{u'^2}-\overline{v'^2}+\frac{\overline{\Phi_z'^2}}{N^2}\right)dy \\
&= \rho_0\sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}2^{n-1}n!\sqrt{\pi}\beta|\tilde{m}|N\left[\frac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2}+\frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2}-\frac{1}{\beta|\tilde{m}|N}\right] \\
&= \rho_0\sqrt{\frac{N}{\beta|\tilde{m}|}}\frac{\hat{v}_0^2}{2}2^{n-1}n!\sqrt{\pi}\frac{k}{\hat{\omega}}\frac{2\hat{\omega}N^2k+N^2\beta}{|\tilde{m}^2|\hat{\omega}^2-N^2k^2}.
\end{aligned} \tag{B1}$$

The derivation of (B1) is given in the following.

$$
\begin{aligned}
\tfrac{n+1}{(|\tilde{m}|\hat{\omega}-Nk)^2} \quad & + \frac{n}{(|\tilde{m}|\hat{\omega}+Nk)^2} - \frac{1}{\beta|\tilde{m}|N} \\
= \quad & \frac{(|\tilde{m}|^2\hat{\omega}^2+N^2k^2)(2n+1) - |\tilde{m}|^2\hat{\omega}^2 - N^2k^2 + (|\tilde{m}|\hat{\omega}+Nk)^2}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
- \quad & \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)\{k\hat{\omega}^{-1}\beta N^2 + (2n+1)\beta|\tilde{m}|N\}}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
= \quad & \frac{2N^2k^2(2n+1) + 2|\tilde{m}|\hat{\omega}Nk - \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)\beta N^2k}{\beta|\tilde{m}|N\hat{\omega}}}{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
= \quad & \frac{(2N^2k^2 - k\hat{\omega}^{-1}\beta N^2)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2) - 2N^4k^3\hat{\omega}^{-1}\beta + 2|\tilde{m}|^2\hat{\omega}N^2k\beta}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
= \quad & \frac{(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)(2N^2k^2 + k\hat{\omega}^{-1}\beta N^2)}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)^2} \\
= \quad & \frac{k}{\hat{\omega}}\frac{2\hat{\omega}N^2k + N^2\beta}{(\beta|\tilde{m}|N)(|\tilde{m}|^2\hat{\omega}^2-N^2k^2)}.
\end{aligned}
\tag{B2}
$$

Thus, meridional integral of $F_{11}^{(\mathrm{EQ})}$ becomes equal to that of 3D wave activity flux applicable to inertia-gravity waves.

APPENDIX C

Another expression of $\overline{u'\eta'}$

In this section, we introduce another expression of $\overline{u'\eta'}$ without using parcel displacements.

From meridional derivative of (2.1a), meridional derivative of (2.1b), (2.1c), and vertical derivative of (2.1d), perturbation potential vorticity equation is expressed as follows.

$$
\bar{D}q' + \beta v' = 0, \quad q' = v'_x - u'_y + \frac{\beta y}{N^2}\rho_0^{-1}(\rho_0\Phi'_z)_z. \tag{C1}
$$

Substituting (2.13b), (2.14a) and (2.14b) into (C1), perturbation potential vorticity q' is expressed in terms of $\hat{v}_0$ as follows.

$$
\begin{aligned}
v'_x &= ik\hat{v}_0 H_n e^{-Y^2/2}, &\text{(C2a)}\\
-u'_y &= -i\hat{v}_0\sqrt{\frac{\beta|\tilde{m}|}{N}}\sqrt{\beta|\tilde{m}|N}\frac{d}{dY}\left(\frac{1/2H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk}+\frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)e^{-Y^2/2}\\
&= i\hat{v}_0\beta|\tilde{m}|Y\left(\frac{1/2H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk}+\frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)e^{-Y^2/2}\\
&- i\hat{v}_0\beta|\tilde{m}|\left(\frac{(n+1)H_n}{|\tilde{m}|\hat{\omega}-Nk}+\frac{-nH_n+2YnH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)e^{-Y^2/2}, &\text{(C2b)}\\
\frac{\beta y}{N^2}\rho_0^{-1}(\rho_0\Phi'_z)_z &= -i\hat{v}_0\frac{\beta y}{N}|\tilde{m}|^2\sqrt{\frac{\beta N^3}{|\tilde{m}|}}\left(\frac{1/2H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk}-\frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)e^{-Y^2/2}\\
&= -i\hat{v}_0\beta|\tilde{m}|Y\left(\frac{1/2H_{n+1}}{|\tilde{m}|\hat{\omega}-Nk}-\frac{nH_{n-1}}{|\tilde{m}|\hat{\omega}+Nk}\right)e^{-Y^2/2}, &\text{(C2c)}
\end{aligned}
$$

and

$$
\begin{aligned}
q' &= i\hat{v}_0\left\{kH_n-\beta|\tilde{m}|\left(\frac{(n+1)H_n}{|\tilde{m}|\hat{\omega}-Nk}-\frac{nH_n}{|\tilde{m}|\hat{\omega}+Nk}\right)\right\}e^{-Y^2/2}\\
&= i\hat{v}_0\left\{kH_n-\beta|\tilde{m}\frac{(2n+1)Nk+|\tilde{m}|\hat{\omega}}{|\tilde{m}|^2\hat{\omega}^2-N^2k^2}\right\}H_n e^{-Y^2/2}=i\hat{v}_0\frac{\beta}{\hat{\omega}}H_n e^{-Y^2/2}, &\text{(C2d)}
\end{aligned}
$$

where the dispersion relation of equatorial waves (2.13a) is used in the last line. Thus,

$$
\overline{u'\eta'} = -\frac{\overline{u'q'}}{\beta}. \tag{C3}
$$

It should be noted that this expression can be used for all equatorial waves.

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